IMO WINTER AND SUMMER CAMPS 2007
INEQUALITIES

also 2004 and 2005

A BRIEF SUMMARY OF IMPORTANT RESULTS (WINTER CAMP).

1. **The triangle inequality**
   If a, b, c are real numbers, then $\big||a-c|-|b-c|\big| \le |a-b| \le \big||a-c|+|b-c|\big|$.
2. **The harmonic-geometric-arithmetic-quadratic means inequality**
   If $x_1, x_2, x_3, \dots, x_n$ are positive numbers, then
   $$\frac{n}{\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\dots+\frac{1}{x_n}} \le \sqrt[n]{x_1 x_2 x_3 \dots x_n} \le \frac{x_1+x_2+x_3+\dots+x_n}{n} \le \sqrt{\frac{x_1^2+x_2^2+x_3^2+\dots+x_n^2}{n}}$$
   with equality if and only if $x_1 = x_2 = x_3 = \dots = x_n$.
3. **The general means inequality**
   Let $x_1, x_2, x_3, \dots, x_n$ be positive numbers.
   We define $M_r = \left(\frac{x_1^r+x_2^r+x_3^r+\dots+x_n^r}{n}\right)^{1/r}$ for $r \ne 0$ and $M_0 = \sqrt[n]{x_1 x_2 x_3 \dots x_n}$.
   If $r > s$ then $M_r \ge M_s$, with equality if and only if $x_1 = x_2 = x_3 = \dots = x_n$.
4. **The general weighted means inequality**
   Let $x_1, x_2, x_3, \dots, x_n, w_1, w_2, w_3, \dots, w_n$ be positive numbers with $w_1 + w_2 + w_3 + \dots + w_n = 1$.
   We define $WM_r = \left(w_1 x_1^r + w_2 x_2^r + w_3 x_3^r + \dots + w_n x_n^r\right)^{1/r}$ for $r \ne 0$ and $WM_0 = x_1^{w_1} x_2^{w_2} x_3^{w_3} \dots x_n^{w_n}$.
   If $r > s$ then $WM_r \ge WM_s$, with equality if and only if $x_1 = x_2 = x_3 = \dots = x_n$.
5. **The Minkowski inequality**
   If $x_1, x_2, x_3, \dots, x_n, y_1, y_2, y_3, \dots, y_n$ are all ≥ 0 and $p \ge 1$, then
   $$\left(\sum_{k=1}^n (x_k + y_k)^p\right)^{1/p} \le \left(\sum_{k=1}^n x_k^p\right)^{1/p} + \left(\sum_{k=1}^n y_k^p\right)^{1/p}$$
   with equality if and only if there exists λ such that $y_k = \lambda x_k$ for $k = 1, 2, 3, \dots, n$.
   The inequality is reversed if $0 < p < 1$.
6. **The Cauchy-Schwarz inequality**
   If $v_1, v_2, v_3, \dots, v_n$ and $w_1, w_2, w_3, \dots, w_n$ are real numbers, then
   $|v_1 w_1 + v_2 w_2 + v_3 w_3 + \dots + v_n w_n| \le \sqrt{v_1^2+v_2^2+v_3^2+\dots+v_n^2}\,\sqrt{w_1^2+w_2^2+w_3^2+\dots+w_n^2}$,
   with equality if and only if there exists λ such that $w_k = \lambda v_k$ for $k = 1, 2, 3, \dots, n$.
7. **The Hölder inequality**
   If $x_1, x_2, x_3, \dots, x_n, y_1, y_2, y_3, \dots, y_n, p, q$ are all ≥ 0 and $p + q = 1$, then
   $$\sum_{i=1}^n x_i^p y_i^q \le \left(\sum_{i=1}^n x_i\right)^p \left(\sum_{i=1}^n y_i\right)^q$$
   with equality if and only if there exists λ such that $y_k = \lambda x_k$ for $k = 1, 2, 3, \dots, n$.
8. **The rearrangement inequality**
   Suppose that $x_1 \le x_2 \le x_3 \le \dots \le x_n$ and $y_1 \le y_2 \le y_3 \le \dots \le y_n$, and let $z_1, z_2, z_3, \dots, z_n$ be any permutation of the numbers $y_1, y_2, y_3, \dots, y_n$, then
   $$\sum_{i=1}^n x_i y_{n+1-i} \le \sum_{i=1}^n x_i z_i \le \sum_{i=1}^n x_i y_i .$$

9. **The Chebyshev inequality**
Suppose that $0 \le x_1 \le x_2 \le x_3 \le \ldots \le x_n$ and $0 \le y_1 \le y_2 \le y_3 \le \ldots \le y_n$, then
$$\sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i \le n \sum_{i=1}^{n} x_i y_i .$$

10. **The Schur inequality**
Suppose that $x \ge 0$, $y \ge 0$ and $z \ge 0$, then for any $r > 0$
$$\sum_{cyclic} x^r (x-y)(x-z) \ge 0$$
with equality if only if $x = y = z$ or if two of them are equal and the other is zero.

11. **The Muirhead inequality**
Suppose that $a_1 \ge a_2 \ge a_3 \ge 0$, $b_1 \ge b_2 \ge b_3 \ge 0$, $a_1 \ge b_1$, $a_1 + a_2 \ge b_1 + b_2$ and $a_1 + a_2 + a_3 = b_1 + b_2 + b_3$, then for any $x > 0$, $y > 0$ and $z > 0$
$$\sum_{sym} x^{a_1} y^{a_2} z^{a_3} \ge \sum_{sym} x^{b_1} y^{b_2} z^{b_3}$$
with equality if and only if $x = y = z$.

EXERCISES (WINTER CAMP).

1. Prove that for any positive a, b and c, $(a+b)(b+c)(a+c) \ge 8abc$.
2. Prove that for any positive a, b and c, if $(1+a)(1+b)(1+c) = 8$ then $abc \le 1$.
3. Prove that for any positive a, b and c, if $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$ then $(a-1)(b-1)(c-1) \ge 8$.
4. Prove that for $a, b, c > 0$, if $(a \sin\theta)^2 + (b \cos\theta)^2 < c^2$ then $a \sin^2\theta + b \cos^2\theta < c$.
5. If a, b and c are positive numbers, what is the minimum possible value of the expression
$$\frac{1+a+2b+3c}{\left(1+\sqrt[3]{a}+2\sqrt[3]{b}+3\sqrt[3]{c}\right)^3} ?$$
What are the values of a, b and c for which the minimum value is reached ?
6. What is the maximum possible value of the expression $\dfrac{1+a+2b+3c}{\sqrt{1+2(a^2+b^2+c^2)}}$?
What are the values of a, b and c for which the maximum value is reached ?
7. Find the volume of the largest rectangular box that fits inside the ellipsoid $x^2 + 3y^2 + 9z^2 = 9$, with faces parallel to the coordinate planes.
8. Prove that for $a, b, c, d > 0$, $\dfrac{(a^2+b^2+c^2+d^2)^3}{(abc+abd+acd+bcd)^2} \ge 4$.
9. Prove each of the following inequalities.
a) If $0 \le x \le \pi/2$ then $2x \le \pi \sin x \le \pi x$. (Jordan)
b) If $x > -1$ and $0 < r < 1$, then $(1+x)^r \le 1 + rx$. (Bernoulli)
c) If a, b, p, q are all positive and $p + q = 1$, then $ab \le p a^{1/p} + q b^{1/q}$. (Young)
d) If a, b, c are all positive, then $\dfrac{a}{b+c} + \dfrac{b}{a+c} + \dfrac{c}{a+b} \ge \dfrac{3}{2}$. (Nesbitt)
10. Suppose that there is a triangle whose sides have lengths a, b and c. Prove that there is a triangle whose sides have lengths $\dfrac{a^2+ab+ac+bc}{2a+b+c}$, $\dfrac{ab+ac+b^2+bc}{a+2b+c}$ and $\dfrac{ab+ac+bc+c^2}{a+b+2c}$.
11. Prove the rearrangement inequality.
12. Prove the Chebyshev inequality.

13. Let $n > 3$ be an integer and let $x_1, x_2, x_3, \dots, x_n$ be positive numbers such that $x_1^2 + x_2^2 + \dots + x_n^2 = 1$.

Prove that $\dfrac{x_1}{1+x_2^2} + \dfrac{x_2}{1+x_3^2} + \dots + \dfrac{x_n}{1+x_1^2} \geq \dfrac{4}{5}\left(x_1\sqrt{x_1} + x_2\sqrt{x_2} + \dots + x_n\sqrt{x_n}\right)^2$.

14. Let $x_1, x_2, x_3, \dots, x_n$ be arbitrary real numbers. Prove the inequality

$$\frac{x_1}{1+x_1^2} + \frac{x_2}{1+x_1^2+x_2^2} + \dots + \frac{x_n}{1+x_1^2+\dots+x_n^2} < \sqrt{n}\ .$$

15. Find all positive integers n such that $3^n + 4^n + \dots + (n+2)^n = (n+3)^n$.

16. Find a solution to the system $\begin{cases} a + b + c + d + e = 8 \\ a^2 + b^2 + c^2 + d^2 + e^2 = 16 \end{cases}$ for which the value of e is the maximum possible.

17. Let $x_i > 0$, $x_1 + x_2 + x_3 + \dots + x_n = 1$ and let s be the greatest of the numbers

$$\frac{x_1}{1+x_1}, \frac{x_2}{1+x_1+x_2}, \frac{x_3}{1+x_1+x_2+x_3}, \dots, \frac{x_n}{1+x_1+x_2+\dots+x_n}$$

Find the smallest value for s. Find the values of $x_1, x_2, x_3, \dots, x_n$ for which s reaches its minimum.

18. CMO 2000. P5

Suppose that the real numbers $a_1, a_2, \dots, a_{100}$ satisfy $a_1 \geq a_2 \geq \dots \geq a_{100} \geq 0$, $a_1 + a_2 \leq 100$ and $a_3 + a_4 + \dots + a_{100} \leq 100$. Determine the maximum value of $a_1^2 + a_2^2 + \dots + a_{100}^2$ and find all possible sequences $a_1, a_2, \dots, a_{100}$ which achieve this maximum.

19. CMO 2002. P3

Prove that for all positive real numbers a, b and c,

$$\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} \geq a+b+c\ ,$$

and determine when equality occurs.

20. APMO 2004. P5

Let a, b and c be positive real numbers. Prove that

$$(a^2+2)(b^2+2)(c^2+2) \geq 9(ab+bc+ca)$$

21. APMO 2005. P2

Let a, b and c be positive real numbers such that $abc = 8$. Prove that

$$\frac{a^2}{\sqrt{(1+a^3)(1+b^3)}} + \frac{b^2}{\sqrt{(1+b^3)(1+c^3)}} + \frac{c^2}{\sqrt{(1+c^3)(1+a^3)}} \geq \frac{4}{3}$$

22. IMO 1975. A1

Let $x_1, x_2, x_3, \dots, x_n$ and $y_1, y_2, y_3, \dots, y_n$ be real numbers such that $x_1 \leq x_2 \leq \dots \leq x_n$ and $y_1 \leq y_2 \leq \dots \leq y_n$. Prove that, if $z_1, z_2, z_3, \dots, z_n$ is any permutation of $y_1, y_2, y_3, \dots, y_n$, then

$$\sum_{i=1}^{n} (x_i - y_i)^2 \leq \sum_{i=1}^{n} (x_i - z_i)^2\ .$$

23. IMO 1978. B2

Let $a_1, a_2, a_3, \dots, a_n$ be a sequence of distinct positive integers. Prove that, for all natural numbers n,

$$\sum_{k=1}^{n} \frac{a_k}{k^2} \geq \sum_{k=1}^{n} \frac{1}{k}.$$

24. IMO 1984. A1

Prove that $0 \leq xy + yz + zx - 2xyz \leq 7/27$, where x, y, z are non-negative real numbers such that $x + y + z = 1$.

SOME RECENT IMO PROBLEMS.

25. IMO 2006. A3

Determine the least real number M such that the inequality

$$|ab(a^2-b^2)+bc(b^2-c^2)+ca(c^2-a^2)| \le M(a^2+b^2+c^2)^2$$

holds for all real numbers a, b and c.

26. IMO 2005. A3

Let x, y and z be positive real numbers such that $xyz \ge 1$. Prove that

$$\frac{x^5-x^2}{x^5+y^2+z^2}+\frac{y^5-y^2}{x^2+y^5+z^2}+\frac{z^5-z^2}{x^2+y^2+z^5} \ge 0$$

27. IMO 2004. B1.

Let $n \ge 3$ be an integer. Let $t_1, t_2, t_3, \dots, t_n$ be positive real numbers such that

$$n^2+1 > (t_1+t_2+\dots+t_n)\left(\frac{1}{t_1}+\frac{1}{t_2}+\dots\frac{1}{t_n}\right)$$

Show that t_i, t_j and t_k are side lengths of a triangle for all i, j and k with $1 \le i < j < k \le n$.

28. IMO 2003. B2.

Let $n > 2$ be a positive integer and let $x_1, x_2, \dots, x_n$ be real numbers with $x_1 \le x_2 \le \dots \le x_n$.

a) Show that $\left(\sum_{i=1}^{n}\sum_{j=1}^{n}|x_i-x_j|\right)^2 \le \frac{2}{3}(n^2-1)\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i-x_j)^2$.

b) Show that equality holds if and only if $x_1, x_2, \dots, x_n$ is an arithmetic progression.

29. IMO 2001. A2.

Let a, b and c be positive real numbers. Prove that $\dfrac{a}{\sqrt{a^2+8bc}}+\dfrac{b}{\sqrt{b^2+8ca}}+\dfrac{c}{\sqrt{c^2+8ab}} \ge 1$.

30. IMO 2000. A2.

Let a, b and c be positive real numbers such that $abc = 1$.
Prove that $(a-1+1/b)(b-1+1/c)(c-1+1/a) \le 1$.

31. IMO 1999. A2.

Let $n \ge 2$ be a fixed integer.

a) Determine the least constant C such that the inequality $\sum_{1\le i<j\le n} x_i x_j(x_i^2+x_j^2) \le C(\sum_{1\le i\le n} x_i)^4$ holds for all real numbers $x_1, x_2, \dots, x_n \ge 0$.

b) For this constant C, determine when the equality holds.

32. IMO 1997. A3.

Let $x_1, x_2, \dots, x_n$ be real numbers satisfying the conditions $|x_1+x_2+\dots+x_n| = 1$ and $|x_i| \le \dfrac{n+1}{2}$ for $i = 1, 2, \dots, n$. Show that there exists a permutation $y_1, y_2, \dots, y_n$ of $x_1, x_2, \dots, x_n$ such that

$$|y_1+2y_2+\dots+ny_n| \le \frac{n+1}{2}.$$

A BRIEF SUMMARY OF IMPORTANT RESULTS (SUMMER CAMP).

1. **The Hölder inequality (generalized)**
   If $x_{11}, x_{12}, \ldots, x_{1n}, x_{21}, x_{22}, \ldots, x_{2n}, \ldots x_{m1}, x_{m2}, \ldots, x_{mn}$, are all ≥ 0 and $p_1, p_2, \ldots, p_m$, are all > 0, with $\sum_{i=1}^{m} p_i = 1$, then
   $$\prod_{i=1}^{m} (\sum_{j=1}^{n} x_{ij})^{p_i} \geq \sum_{j=1}^{n} (\prod_{i=1}^{m} x_{ij}^{p_i}) .$$
   Notice that this is also a generalization of the Cauchy-Schwarz inequality.
2. **The Muirhead inequality (generalized)**
   Suppose that the numbers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are such that $a_1 \geq a_2 \geq \ldots \geq a_n \geq 0$, $b_1 \geq b_2 \geq \ldots \geq b_n \geq 0$, $\sum_{i=1}^{k} a_i \gneqq \sum_{i=1}^{k} b_i$ for $1 \leq k \leq n-1$, and $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i$.
   Then, then for any positive numbers $x_1, x_2, \ldots, x_n$
   $$\sum_{sym} x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \geq \sum_{sym} x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n} .$$
3. **The Jensen inequality**
   Suppose that $f : [a, b] \to \mathbb{R}$ is such that $\lambda f(x) + (1-\lambda) f(y) \geq f(\lambda x + (1-\lambda) y)$ for all $x \in [a, b]$, $y \in [a, b]$ and $0 \leq \lambda \leq 1$. Suppose also that the numbers $x_1, x_2, x_3, \ldots, x_n$ and $w_1, w_2, w_3, \ldots, w_n$ are such $x_i \in [a, b]$ and $w_i \geq 0$ for all i, and $\sum_{i=1}^{n} w_i = 1$. Then
   $$\sum_{i=1}^{n} w_i f(x_i) \geq f(\sum_{i=1}^{n} w_i x_i) .$$
   Note: A function $f : [a, b] \to \mathbb{R}$ such that $\lambda f(x) + (1-\lambda) f(y) \geq f(\lambda x + (1-\lambda) y)$ for all $x \in [a, b]$, $y \in [a., b]$ and $0 \leq \lambda \leq 1$ is said to be "convex" over the interval $[a, b]$.
   If $\lambda f(x) + (1-\lambda) f(y) \leq f(\lambda x + (1-\lambda) y)$ for all $x \in [a, b]$, $y \in [a, b]$ and $0 \leq \lambda \leq 1$ then the function f is said to be "concave" over the interval $[a, b]$. In this case Jensen's inequality states that
   $$\sum_{i=1}^{n} w_i f(x_i) \leq f(\sum_{i=1}^{n} w_i x_i).$$

EXERCISES (SUMMER CAMP).

1. Let a, b and c be positive numbers such that $abc = 1$. Prove that
   $$a^2 + b^2 + c^2 \geq a + b + c .$$
2. Let a, b and c be positive numbers such that $a + b + c = 1$. Prove that
   $$a^3 + b^3 + c^3 + 6abc \geq \frac{1}{4} .$$
3. **IMO 1995. A2.**
   Let a, b and c be positive numbers such that $abc = 1$. Prove that
   $$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2} .$$

4. Let a , b and c be positive numbers such that $a+b+c=abc$. Prove that

$$\frac{1}{\sqrt{1+a^2}}+\frac{1}{\sqrt{1+b^2}}+\frac{1}{\sqrt{1+c^2}}\leq\frac{3}{2}.$$

5. Let a , b , c , m and n be positive numbers . Prove that (tr2001 p22)

$$\frac{a}{bm+cn}+\frac{b}{cm+an}+\frac{c}{am+bn}\leq\frac{3}{m+n}.$$

6. Let a , b , c , d and e be positive numbers . Prove that

$$\sum_{cyc}\frac{a}{b+2c+3d+4e}\geq\frac{1}{2}.$$

7. Let n be a positive integer and let $a_1, a_2, \ldots, a_n$ be positive real numbers. Prove that

$$\frac{a_1^4}{a_1^2+a_2^2}+\frac{a_2^4}{a_2^2+a_3^2}+\ldots+\frac{a_n^4}{a_n^2+a_1^2}\geq\frac{1}{2n}.$$

8. Let a , b , c and d be non negative numbers such that $ab+bc+cd+da=1$. Prove that

$$\sum_{cyc}\frac{a^3}{b+c+d}\geq\frac{1}{3}.$$

9. **IMO 1974. B3.**
Let a , b , c and d be positive numbers. Determine all the possible values of the expression

$$\sum_{cyc}\frac{a}{a+b+d}.$$

10. Let a , b and c be positive numbers. Prove that

[illegible] use $a^3+b^3 \geq ab(a+b)$)

$$\sum_{cyc}\frac{1}{a^3+b^3+abc}\leq\frac{1}{abc}.$$

11. Let a , b and c be positive numbers. Prove that

$$\sum_{cyc}\frac{(b+c-a)^2}{(b+c)^2+a^2}\geq\frac{3}{5}.$$

12. Let $P(x)$ be a polynomial with non negative coefficients and let a , b and c be non negative numbers such that $P(a^3)\leq 2^3$, $P(b^3)\leq 3^3$ and $P(c^3)\leq 7^3$. Prove that

$$P(abc)\leq 4^3.$$

13. Let a , b and c be positive numbers such that $abc=1$. Prove that

$$\sum_{cyc}\frac{1}{(a+1)^2+b^2+1}\geq\frac{1}{2}.$$

14. Let a , b and c be positive numbers such that $abc=1$. Prove that

$$\sum_{cyc}\frac{1}{(a+1)^2+b^2+1}\geq\frac{1}{2}.$$

15. Let a , b and c be positive numbers such that $ab+bc+ca=1$. Prove that

$$\sum_{cyc}\sqrt[3]{\frac{1}{a}+6b}\leq\frac{1}{abc}.$$

16. Let a, b and c be positive numbers. Prove that

$$\sum_{cyc} \frac{1}{a(1+b)} \geq \frac{3}{1+abc},$$

with equality if and only if $a=b=c=1$.

17. Let a, b and c be positive numbers. Prove that

$$\sum_{cyc} \frac{\sqrt{b+c}}{a} \geq \frac{4(a+b+c)}{\sqrt{(a+b)(b+c)(c+a)}}.$$

18. Let $x_1, x_2, \ldots, x_n$ be real numbers such that $-1 \leq x_i \leq 1$ for $i=1,2,\ldots,n$ and $\sum_{i=1}^{n} x_i^3 = 0$.

Prove that $\sum_{i=1}^{n} x_1 \leq \frac{n}{3}$.

19. Let $x_1, x_2, \ldots, x_n$ be positive real numbers such that $\sum_{i=1}^{n} x_i^2 = 1$, where $n \geq 2$. Determine the smallest

possible value of the sum $\sum_{cyc} \frac{x_1^5}{x_2+x_3+\ldots+x_n}$.

20. Let $x_1, x_2, \ldots, x_n$ be positive real numbers such that $\sum_{i=1}^{n} x_i^{-1} = n$. Determine the smallest possible value

of the sum $\sum_{i=1}^{n} \frac{x_i^i}{i}$.

21. Find the minimum value of c such that $\sum_{i=1}^{n} \sqrt{x_i} \geq c\sqrt{\sum_{i=1}^{n} x_i}$, for any n and any non negative numbers

$x_1, x_2, \ldots, x_n$ which satisfy the condition $x_{i+1} \geq \sum_{k=1}^{i-1} x_k$, for $i=1,2,\ldots,n-1$.

22. Let $x_1, x_2, \ldots, x_n$ be positive real numbers. Prove that

$$\sum_{cyc} \frac{x_1^3}{x_1^2+x_1x_2+x_2^2} \geq \frac{\sum_{i=1}^{n} x_i}{3}.$$

23. IMO 1999. A2.

Find the minimum value of c such that $\sum_{1\leq i<j\leq n} x_i x_j (x_i^2+x_j^2) \leq c(\sum_{i=1}^{n} x_i)^4$, for any $n \geq 2$ and any non

negative numbers $x_1, x_2, \ldots, x_n$.

24. Let $x_1, x_2, \ldots, x_n$ be positive real numbers. Prove that

$$\sum_{cyc} \frac{x_1^2}{x_1+x_2} \geq \frac{\sum_{i=1}^{n} x_i}{2}$$